Composite Structure with Highly Contrasting Conductivities:
Homogenization of Hyperbolic Equation

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A Brief Introduction to Homogenization
What is Homogenization (Mathematical)?

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Define, for $\alpha, \beta > 0$, the class of matrix functions:

$$E(\Omega) = E(\alpha, \beta, \Omega) = \{A = [a_{ij}(x)] : A \text{ is symmetric and satisfies (1)}\}.$$
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We assume the matrix $A$ satisfies

$$\alpha |\xi|^2 \leq \langle A(x)\xi, \xi \rangle = a_{ij} \xi_i \xi_j \leq \beta |\xi|^2, \forall \xi \in \mathbb{R}^n. \quad (1)$$

The first inequality is nothing but the uniform ellipticity.
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The first inequality is nothing but the uniform ellipticity.

Given an element $A \in E(\Omega)$, associate the PDE operator $A = -\frac{\partial}{\partial x_i}(a_{ij} \frac{\partial}{\partial x_j})$ and introduce the elliptic boundary value problem

$$Au = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega. \quad (2)$$
What is Homogenization (Mathematical)?, Conti..

The aim is to introduce certain convergence in the above class of matrix functions relevant to the homogenization theory.
What is Homogenization (Mathematical)?, Conti..

- The aim is to introduce certain convergence in the above class of matrix functions relevant to the homogenization theory.

- \((G\)-convergence or \(H\)-convergence): We say a family \([a_{ij}]\) \(\varepsilon \to 0\), \(H\)-converges to \(a_{ij}^*\) as \(\varepsilon \to 0\) if
  
  \[i) \quad u_\varepsilon \rightharpoonup u \text{ in } H^1_0(\Omega) \text{ weak}\]
  
  \[ii) \quad a^\varepsilon_{ij}(x) \frac{\partial u_\varepsilon}{\partial x_j} \rightharpoonup a^*_{ij}(x) \frac{\partial u}{\partial x_j} \text{ in } L^2(\Omega) \text{ weak.}\]
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** (G- convergence or H - convergence): We say a family \{[a_{ij}^\varepsilon]\}_{\varepsilon>0}, H\text{-converges to } [a_{ij}^*]\text{ as } \varepsilon \to 0\text{ if}

\begin{align*}
  i) & \quad u_\varepsilon \rightharpoonup u \text{ in } H^1_0(\Omega) \text{ weak} \\
  ii) & \quad a_{ij}^\varepsilon(x) \frac{\partial u_\varepsilon}{\partial x_j} \rightharpoonup a_{ij}^*(x) \frac{\partial u}{\partial x_j} \text{ in } L^2(\Omega) \text{ weak.}
\end{align*}

Here \( u^\varepsilon, u \) are, respectively, the solution of (2) corresponding to the operators \( A^\varepsilon, A^* \) and we write

\[
[a_{ij}^\varepsilon] \underset{H}{\rightharpoonup} [a_{ij}^*] \text{ or simply } A^\varepsilon \underset{H}{\rightharpoonup} A^*.
\]
So in a nutshell, one has a family of differential operators and we would like to obtain the limit operator. In some sense the limit behaviour of the differential operators.
What is Homogenization (Mathematical)?, Conti..

♣ So in a nutshell, one has a family of differential operators and we would like to obtain the limit operator. In some sense the limit behaviour of the differential operators.

♣ The differential operators need not be elliptic, it can come from many other situations, like parabolic, hyperbolic based on applications. But studying for elliptic operators is more fundamental even in other applications.
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Why do we need to study such a special convergence? Why it is different from other asymptotic analysis and a new name homogenization? For this one need to understand the physical applications of homogenization.
What is Homogenization (Physical)?

It is the study of macroscopic and/or bulk behavior of solutions to partial differential or other type of equations posed on a heterogeneous domain/media, where the heterogeneities are present at microscopic scale $\varepsilon$. 
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♣ It is the study of macroscopic and/or bulk behavior of solutions to partial differential or other type of equations posed on a heterogeneous domain/media, where the heterogeneities are present at microscopic scale $\varepsilon$.

♣ The heterogeneities can be in the form of fine mixing of two or more materials with different physical properties in the domain or due to singularities in the domain in the form of pores, granules etc.
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- analysis of vibrations of thin structures.
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- study of flow in porous media (flow of oil, water through subsurface, pollution of ground water, flow of resins and polymers in moulds etc.)
- analysis of vibrations of thin structures.
- homogenization with oscillating (rough) boundary.
Few Sample Composites
Few Sample Composites; conti...

Plywood

Concrete
Few Sample Composites; conti...

Figure 21. Microstructures of heterogeneous materials: aluminum with aluminum oxide reinforcement (left) and SiC-Titanium composite (right).
Figure 1.1. The two-dimensional microstructure of Larsen, Sigmund, and Bouwstra (1997), which will expand laterally when stretched longitudinally. Here the black region is relatively stiff and is surrounded by a void or very compliant material.
Sample Composites; Conti..

Figure 22.1. A second-rank laminate polycrystal that has the largest effective conductivity amongst all isotropic conducting polycrystals. The volume fractions $f$ and $f' = 1 - f$ are chosen so that the conductivity in the $x_3$ direction is the arithmetic average $(\lambda_1 + \lambda_2 + \lambda_3)/3$. The conductivities in the other two directions are then also the arithmetic average. After Avellaneda, Cherkaev, Lurie, and Milton (1988).
Sample Composites, Conti..
Periodically Perforated Domain
Oscillating Boundary: Sample Domains
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♦ The problem posed on the heterogeneous domain has a unique solution.

♦ Heterogeneities cause the solution to develop high frequency oscillations.

♦ So a direct numerical study will not be able to capture either the bulk behavior or the oscillations present in the solutions which are at a microscopic level.
Periodic Oscillations in Heterogeneities
What is the way out?

Do an asymptotic analysis as $\varepsilon \to 0$, that is we try to approximate the solution $u_\varepsilon$ corresponding to the heterogeneous domain, for small $\varepsilon > 0$, by the solution of a homogenized problem in which the small parameter do not appear.
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The mathematical problem is:
- to identify the homogenized equation
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• to prove convergence of $u_\varepsilon$ corresponding to the heterogeneous equation to that of the homogeneous equations as $\varepsilon \to 0$. 
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- to capture the oscillations in $u_\varepsilon$ (known as correctors in the literature)
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**The mathematical problem is:**
- to identify the homogenized equation
- to prove convergence of $u_\varepsilon$ corresponding to the heterogeneous equation to that of the homogeneous equations as $\varepsilon \to 0$.
- to capture the oscillations in $u_\varepsilon$ (known as **correctors** in the literature)
- the solution together with the correctors can be used (and also can be computed numerically) to get good approximation to the original solution.
Various Methods

- Formal asymptotic expansion
- Energy method via test functions
- Compensated Compactness
- Gamma Convergence
- Two Scale (Multi-scale) Convergence
- Fourier (Bloch wave) method
- Unfolding method
Composite Structure with Two High Contrasting Materials

Composite material \( \Omega_{\varepsilon} = B_{\varepsilon} \cup M_{\varepsilon} \) with:

- Soft inclusions, \( B_{\varepsilon} \)
- Stiff components, \( M_{\varepsilon} \)
Hyperbolic Equation

\[
(P_\varepsilon) \quad \begin{cases}
L_\varepsilon u_\varepsilon := u''_\varepsilon - \text{div} \left( a_\varepsilon(x) A \left( x, \frac{x}{\varepsilon} \right) \nabla u_\varepsilon \right) + u_\varepsilon = f_\varepsilon \\
\text{in } \Omega_T = (0, T) \times \Omega, \\
u_\varepsilon = 0 \text{ on } \partial\Omega_T = (0, T) \times \Gamma, \quad u_\varepsilon(0) = u^0_\varepsilon, \quad u'_\varepsilon(0) = u^1_\varepsilon.
\end{cases}
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Hyperbolic Equation

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\begin{cases}
L_\varepsilon u_\varepsilon := u_\varepsilon'' - \text{div} \left( a_\varepsilon(x) A(x, \frac{x}{\varepsilon}) \nabla u_\varepsilon \right) + u_\varepsilon = f_\varepsilon \\
\text{in } \Omega_T = (0, T) \times \Omega,
\end{cases}
\]

\[
\begin{align*}
L_\varepsilon u_\varepsilon &= 0 \text{ on } \partial \Omega_T = (0, T) \times \Gamma, \\
u_\varepsilon(0) &= u_\varepsilon^0, \quad u_\varepsilon'(0) = u_\varepsilon^1.
\end{align*}
\]

- The coefficients \( a_\varepsilon \) takes the form \( a_\varepsilon(x) = \alpha_\varepsilon^2 \chi_{B_\varepsilon} + \chi_{M_\varepsilon} \) and \( A \) is uniformly elliptic.
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- Here \( \alpha_\varepsilon \) is a small parameter which goes to zero. Let \( \alpha = \lim_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\varepsilon} \).
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- 3 cases; \( \alpha = 0, \alpha = +\infty \) and \( 0 < \alpha < \infty \) which is the critical case. We mainly concentrate on critical case and take \( \alpha_\varepsilon = \varepsilon \).
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- 3 cases; $\alpha = 0$, $\alpha = +\infty$ and $0 < \alpha < \infty$ which is the critical case. We mainly concentrate on critical case and take $\alpha_\varepsilon = \varepsilon$.

- Hence $B_\varepsilon$ is the soft inclusions and $M_\varepsilon$ is stiff material part.
Hyperbolic Equation

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- Here \( \alpha_\varepsilon \) is a small parameter which goes to zero. Let \( \alpha = \lim_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\varepsilon} \).
- 3 cases; \( \alpha = 0 \), \( \alpha = +\infty \) and \( 0 < \alpha < \infty \) which is the critical case. We mainly concentrate on critical case and take \( \alpha_\varepsilon = \varepsilon \).
- Hence \( B_\varepsilon \) is the soft inclusions and \( M_\varepsilon \) is stiff material part.
- We wish to study the limiting behavior of the solution.
Assumptions on the Data

\[(A1) \quad \int_\Omega |u_\varepsilon^0|^2 + \int_{B_\varepsilon} \alpha_\varepsilon^2 |\nabla u_\varepsilon^0|^2 + \int_{M_\varepsilon} |\nabla u_\varepsilon^0|^2 + \int_{\Omega} |u_\varepsilon^1|^2 < \infty\]
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(A2) \[ \|f_\varepsilon\|_{L^2(\Omega_T)} \leq C, \]
Assumptions on the Data

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(A2) \[ \|f_\varepsilon\|_{L^2(\Omega_T)} \leq C, \]

The assumption (A1) is natural based on energy estimates which we are going to present it shortly. From the second assumption, we get

\[ f_\varepsilon \rightharpoonup f \text{ weakly in } L^2(\Omega_T) \tag{3} \]

\[ f_\varepsilon \overset{2-s}{\rightharpoonup} f_0(t, x, y) \text{ in } L^2(\Omega_T) \text{ and } f(t, x) = \int f_0(t, x, y) dy \tag{4} \]
Assumptions on the Data

\((A1)\) \[ \int_{\Omega} |u_\varepsilon^0|^2 + \int_{B_\varepsilon} \alpha_\varepsilon^2 |\nabla u_\varepsilon^0|^2 + \int_{M_\varepsilon} |\nabla u_\varepsilon^0|^2 + \int_{\Omega} |u_\varepsilon^1|^2 < \infty \]

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\[ f_\varepsilon \rightharpoonup f \text{ weakly in } L^2(\Omega_T) \quad (3) \]

\[ f_\varepsilon \rightharpoonup f_0(t, x, y) \text{ in } L^2(\Omega_T) \text{ and } f(t, x) = \int f_0(t, x, y) \, dy \quad (4) \]

- The second convergence is known as two-scale convergence which we use it for our homogenization problem.
Two Scale Convergence

Definition (Two-scale convergence)

A sequence of functions \( \{v_\varepsilon\} \) in \( L^2(\Omega_T) \) is said to two-scale converge to a limit \( v \in L^2(\Omega_T \times Y) \) (denoted as \( v_\varepsilon \overset{2s}{\rightharpoonup} v \)) if

\[
\int_{\Omega_T} v_\varepsilon \phi \left( t, x, \frac{x}{\varepsilon} \right) \, dx \, dt \rightarrow \int_{\Omega_T} \int_Y v(t, x, y) \phi(t, x, y) \, dy \, dx \, dt
\]

For all \( \phi \in L^2(\Omega_T; C_\#(Y)) \).
Two Scale Convergence

**Definition (Two-scale convergence)**

A sequence of functions $\{v_\varepsilon\}$ in $L^2(\Omega_T)$ is said to two-scale converge to a limit $v \in L^2(\Omega_T \times Y)$ (denoted as $v_\varepsilon \overset{2s}{\rightharpoonup} v$) if

$$
\int_{\Omega_T} v_\varepsilon \phi \left( t, x, \frac{x}{\varepsilon} \right) \, dx \, dt \to \int_{\Omega_T} \int_Y v(t, x, y) \phi(t, x, y) \, dy \, dx \, dt
$$

For all $\phi \in L^2(\Omega_T; C_\#(Y))$.

Further, if $v_0$ is the weak limit of $\{v_\varepsilon\}$ in $L^2(\Omega_T)$, then

$$
v_0(t, x) = \int_Y v(t, x, y) \, dy.
$$
We have the following compactness theorem.

**Theorem (Compactness)**

For any bounded sequence $v_\varepsilon$ in $L^2(\Omega_T)$, there exist a subsequence and $v \in L^2(\Omega_T \times Y)$ such that, $v_\varepsilon$ two-scale converges to $v$ along the subsequence.

Also, if $v_\varepsilon$ is bounded in $L^2(0, T; H^1(\Omega))$, then $v$ is independent of $y$ and is in $L^2(0, T; H^1(\Omega))$, and there exists a $v_1 \in L^2(\Omega_T; H_\#^1(Y))$ such that, up to a subsequence, $\nabla v_\varepsilon$ two-scale converges to $\nabla v + \nabla_y v_1$. □
Energy of the System

\[ E_\varepsilon(t) = \frac{1}{2} \left\{ \int_\Omega |u'_\varepsilon(t)|^2 + \int_\Omega |u_\varepsilon(t)|^2 + \int_\Omega \alpha_\varepsilon^2 \chi_{B_\varepsilon} |\nabla u_\varepsilon(t)|^2 + \int_\Omega \chi_{M_\varepsilon} |\nabla u_\varepsilon(t)|^2 \right\} \]

\[ = E_1^\varepsilon(t) + E_2^\varepsilon(t) + E_3^\varepsilon(t) + E_4^\varepsilon(t) \]
Energy of the System

\[ E_\varepsilon(t) = \frac{1}{2} \left\{ \int_\Omega |u'_\varepsilon(t)|^2 + \int_\Omega |u_\varepsilon(t)|^2 \right. \]
\[ + \left. \int_\Omega \alpha^2_\varepsilon \chi_{B_\varepsilon} |\nabla u_\varepsilon(t)|^2 + \int_\Omega \chi_{M_\varepsilon} |\nabla u_\varepsilon(t)|^2 \right\} \]
\[ = E^1_\varepsilon(t) + E^2_\varepsilon(t) + E^3_\varepsilon(t) + E^4_\varepsilon(t) \]  

Proposition

There exists a constant \( C > 0 \) independent of \( \varepsilon \) such that

\[ E_\varepsilon(t) < C \]
Apriori Estimates

- The above proposition will give us the following estimates

**Proposition**

There exists a constant $C > 0$ independent of $\varepsilon$ such that

\[
\begin{align*}
\| u_\varepsilon \|_{L^\infty_t L^2_x} & \leq C, \\
\| u'_\varepsilon \|_{L^\infty_t L^2_x} & \leq C, \\
\| \alpha_\varepsilon \nabla u_\varepsilon \|_{L^\infty_t L^2_x(B_\varepsilon)} & \leq C, \\
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- Thus, we have the correct \( L^2 \) estimates for the solution and its time-derivative. Also gradient estimate in the stiff part.
Apriori Estimates

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**Proposition**

There exists a constant $C > 0$ independent of $\varepsilon$ such that

\[
\begin{align*}
\|u_\varepsilon\|_{L_t^\infty L_x^2} &\leq C, \\
\|u'_\varepsilon\|_{L_t^\infty L_x^2} &\leq C, \\
\|\alpha_\varepsilon \nabla u_\varepsilon\|_{L_t^\infty L_x^2(B_\varepsilon)} &\leq C, \\
\|\nabla u_\varepsilon\|_{L_t^\infty L_x^2(M_\varepsilon)} &\leq C.
\end{align*}
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- Thus, we have the correct $L^2$ estimates for the solution and its time-derivative. Also gradient estimate in the stiff part.
- The difficulty in this problem: The gradient estimate in the soft inclusions is of order $\varepsilon^{-1}$ and hence, in general $H^1$ estimate is of order $\varepsilon^{-1}$.
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There exists a constant $C > 0$ independent of $\varepsilon$ such that

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\| \alpha_\varepsilon \nabla u_\varepsilon \|_{L^\infty_t L_x^2(B_\varepsilon)} & \leq C, & \| \nabla u_\varepsilon \|_{L^\infty_t L_x^2(M_\varepsilon)} & \leq C.
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$$

Thus, we have the correct $L^2$ estimates for the solution and its time-derivative. Also gradient estimate in the stiff part.

- The difficulty in this problem: The gradient estimate in the soft inclusions is of order $\varepsilon^{-1}$ and hence, in general $H^1$ estimate is of order $\varepsilon^{-1}$.

- This also motivates the assumption (A1) on the initial data.
Extension Operators

- We use the idea from perforated domains, where one extends the solution from the domain part to the perforations via linear operators.
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- Thus idea is to restrict the solution to the **stiff part** of the domain and treating the **soft part** as perforations, extend the solution in a bounded way. This is done using the Extension Lemma.
Extension Operators

- We use the idea from perforated domains, where one extends the solution from the domain part to the perforations via linear operators.

- Thus idea is to restrict the solution to the **stiff part** of the domain and treating the **soft part** as perforations, extend the solution in a bounded way. This is done using the Extension Lemma.

- Then study the convergence of the actual solution and the extended function using two-scale convergence and connect them.
Extension Lemma

Lemma (Cioranescu-Donato)

There exists a linear continuous operator $P^\varepsilon \in \mathcal{L}(L^\infty(0,T;H^k(M_\varepsilon)); L^\infty(0,T;H^k(\Omega)))$, $k = 0, 1$ such that, for some constant $C$ independent of $\varepsilon$: for any $\phi \in L^\infty(0,T;H^k(M_\varepsilon))$;

\[
\begin{align*}
P^\varepsilon \phi &= \phi \quad \text{in} \quad M_\varepsilon \times (0,T) \\
P^\varepsilon \phi' &= (P^\varepsilon \phi)' \quad \text{in} \quad \Omega \times (0,T) \\
\|P^\varepsilon \phi\|_{L^\infty(0,T;L^2(\Omega))} &\leq C \|\phi\|_{L^\infty(0,T;L^2(M_\varepsilon))} \\
\|P^\varepsilon \phi'\|_{L^\infty(0,T;L^2(\Omega))} &\leq C \|\phi'\|_{L^\infty(0,T;L^2(M_\varepsilon))} \\
\|\nabla(P^\varepsilon \phi)\|_{L^\infty(0,T;L^2(\Omega))} &\leq C \|\nabla \phi\|_{L^\infty(0,T;L^2(M_\varepsilon))}
\end{align*}
\]
Estimate and Convergence on extended functions

Let $\tilde{u}_\varepsilon$ be the extension of $u_\varepsilon$ restricted to $M_\varepsilon$, that is $\tilde{u}_\varepsilon = P_\varepsilon(u_\varepsilon|_{M_\varepsilon})$. Then

$$\|\tilde{u}_\varepsilon\|_{L_\infty H^1_0(\Omega)} \leq C, \quad \|\tilde{u}'_\varepsilon\|_{L_\infty L^2_x(\Omega)} \leq C$$

(8)
Let \( \tilde{u}_\varepsilon \) be the extension of \( u_\varepsilon \) restricted to \( M_\varepsilon \), that is \( \tilde{u}_\varepsilon = P^\varepsilon(u_\varepsilon|M_\varepsilon) \). Then

\[
\| \tilde{u}_\varepsilon \|_{L^\infty_t H^1_0(\Omega)} \leq C, \quad \| \tilde{u}'_\varepsilon \|_{L^\infty_t L^2_x(\Omega)} \leq C
\]  

\[=\]

\[
\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \text{ weak* in } L^\infty_t H^1_0(\Omega) \\
\tilde{u}'_\varepsilon \rightharpoonup \tilde{u}' \text{ weak* in } L^\infty_t L^2_x(\Omega)
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\|\tilde{u}_\varepsilon\|_{L^\infty_t H^1_0(\Omega)} \leq C, \quad \|\tilde{u}'_\varepsilon\|_{L^\infty_t L^2_x(\Omega)} \leq C
$$

(8)

$$
\Rightarrow
$$

$$
\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \text{ weak* in } L^\infty_t H^1_0(\Omega)
$$

$$
\tilde{u}'_\varepsilon \rightharpoonup \tilde{u}' \text{ weak* in } L^\infty_t L^2_x(\Omega)
$$

(9)

$$
\Rightarrow
$$

$$
\tilde{u}_\varepsilon \to \tilde{u} \text{ strongly in } L^2(\Omega_T) \text{ and } C([0, T], L^2(\Omega))
$$

(10)
Further Analysis on $u_\varepsilon$

- We assume that $\alpha_\varepsilon = O(\varepsilon)$, i.e., $\frac{\alpha_\varepsilon}{\varepsilon} \to \alpha \in (0, \infty)$
Further Analysis on $u_\varepsilon$

- We assume that $\alpha_\varepsilon = O(\varepsilon)$, i.e., $\frac{\alpha_\varepsilon}{\varepsilon} \to \alpha \in (0, \infty)$

- Since $u_\varepsilon$, $\varepsilon \nabla u_\varepsilon$ are bounded in $L^2(\Omega_T)$, we apply two-scale convergence to get $v_0 = v_0(t, x, y) \in L^2(\Omega_T; H^{1\#}_1(Y))$ such that

$$
\begin{align*}
&u_\varepsilon \rightharpoonup_{2-s} v_0 \text{ in } L^2(\Omega_T) \\
&\varepsilon \nabla u_\varepsilon \rightharpoonup_{2-s} \nabla_y v_0 \text{ in } L^2(\Omega_T) \\
&w'_\varepsilon \rightharpoonup_{2-s} v'_0 \text{ in } L^2(\Omega_T)
\end{align*}
$$

(11)
Further Analysis on $u_\varepsilon$; Conti...

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The difficulty is that we do not have convergence in $H^1$.  

Thus $v_0(t, x, y)$ is constant in $y$ variable in $\Omega \times Y$. In fact, using the identity and 2-scale convergence, we get $v_0(t, x, y) = \tilde{u}(t, x)$ in $\Omega \times M$. For $(t, x, y) \in \Omega_T \times Y$, define $v(t, x, y) := v_0(t, x, y) - \tilde{u}(t, x)$. Thus $v$ vanishes outside $B$.  

Composite Structure: Homogenization  
Cebu, Phillipines: January 15, 2016  
A.K.N/IISc
Further Analysis on $u_\varepsilon$; Conti...

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• But $\nabla u_\varepsilon$ is bounded in $L^2(M_\varepsilon)$, we have $\nabla u_\varepsilon \chi_{M_\varepsilon}$ bounded in $L^2(\Omega_T)$, by suitably applying 2-scale convergence, we have

$$\nabla_y v_0(t, x, y) = 0 \text{ in } \Omega \times M \text{ a.e.}$$
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- For $(t, x, y) \in \Omega_T \times Y$, define
  \[ v(t, x, y) := v_0(t, x, y) - \tilde{u}(t, x). \]

Thus $v$ vanishes outside $B$. 
Further Analysis on $u_\epsilon$; Conti...

We get the following regularity results:

\[ u \in L^\infty(0, T; L^2(\Omega; H^1_0(B))) \quad \text{and} \quad u' \in L^\infty(0, T; L^2(\Omega; L^2(B))) \]

\[ \tilde{u} \in L^\infty_t H^1_0(\Omega), \quad \tilde{u}' \in L^\infty_t (L^2(\Omega)). \]
Further Analysis on $u_\varepsilon$; Conti...

We get the following regularity results:

$$v \in L^\infty(0, T; L^2(\Omega; H^1_0(B))) \text{ and } v' \in L^\infty(0, T; L^2(\Omega; L^2(B)))$$

$$\tilde{u} \in L^\infty_t H^1_0(\Omega), \quad \tilde{u}' \in L^\infty_t (L^2(\Omega)).$$

$$u_\varepsilon \rightharpoonup \frac{2s}{\varepsilon} v_0(t, x, y) = \tilde{u}(t, x) + v(t, x, y).$$
Further Analysis on $u_\varepsilon$; Conti...

We get the following regularity results:

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- The aim, indeed, is to identify $v_0$ which is the two-scale limit of our original problem.
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We get the following regularity results:

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- The aim, indeed, is to identify $v_0$ which is the two-scale limit of our original problem.

- The difficulty is that, in general, we will not have regularity for $v_0$ with respect to the spatial variable $x$. Indeed, we do get regularity with respect to the fast variable $y$. On the other hand, $\tilde{u}$, the limit of the extended function has $H^1$ regularity with respect to $x$. 
For the two-scale limit $v_0$ of the given problem, we have the representation

$$v_0(t, x, y) = \tilde{u}(t, x) + v(t, x, y),$$

where

$$\tilde{u} \in L^2(0, T; H^1_0(\Omega)), \quad v \in L^\infty(0, T; L^2(\Omega; H^1_0(B))).$$
Consolidation of all the Discussion

- For the two-scale limit $v_0$ of the given problem, we have the representation
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  where
  \[ \tilde{u} \in L^2(0, T; H^1_0(\Omega)), \quad v \in L^\infty(0, T; L^2(\Omega; H^1_0(B))). \]

- Here $\tilde{u}$ is the limit of extended function in $L^\infty(0, T; H^1_0(\Omega))$. 
Consolidation of all the Discussion

• For the two-scale limit \( v_0 \) of the given problem, we have the representation

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v_0(t, x, y) = \tilde{u}(t, x) + v(t, x, y),
\]

where

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\tilde{u} \in L^2(0, T; H^1_0(\Omega)), \quad v \in L^\infty(0, T; L^2(\Omega; H^1_0(B))).
\]

• Here \( \tilde{u} \) is the limit of extended function in \( L^\infty(0, T; H^1_0(\Omega)) \).

• First, we will look for a solution in the above form which satisfies a weak formulation.
Limit of $\nabla u_\varepsilon \chi_{M_\varepsilon}$ and $\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon}$

- Clearly $\varepsilon \chi_{B_\varepsilon} \nabla u_\varepsilon \xrightarrow{2s} \chi_B(y) \nabla_y v$. 
Limit of $\nabla u_\varepsilon \chi_{M_\varepsilon}$ and $\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon}$

- Clearly $\varepsilon \chi_{B_\varepsilon} \nabla u_\varepsilon \rightharpoonup^{2s} \chi_B(y) \nabla_y v$.

- We do more analysis on the additional information that $\nabla u_\varepsilon$ is bounded in the stiff part of the material, that is $\nabla u_\varepsilon \chi_{M_\varepsilon}$ is bounded in $L^2(\Omega_T)$. 
Limit of $\nabla u_\varepsilon \chi_{M_\varepsilon}$ and $\varepsilon \nabla u_\varepsilon \chi_{B_\varepsilon}$

- Clearly $\varepsilon \chi_{B_\varepsilon} \nabla u_\varepsilon \overset{2s}{\rightharpoonup} \chi_B(y) \nabla_y v$.

- We do more analysis on the additional information that $\nabla u_\varepsilon$ is bounded in the stiff part of the material, that is $\nabla u_\varepsilon \chi_{M_\varepsilon}$ is bounded in $L^2(\Omega_T)$.

- Let $\nabla u_\varepsilon \chi_{M_\varepsilon} \rightharpoonup K(t, x, y)$ in $L^2(\Omega_T)$. Then, we have

**Proposition**

There exists $u_1(t, x, y) \in L^2(\Omega_T; H^1_\#(M))$ such that

$$[K(t, x, y) - \nabla_x \tilde{u}(t, x)] \chi_M(y) = \nabla_y u_1(t, x, y) \chi_M(y)$$

Further, $u_1$ vanishes outside $M$ satisfies

$$u_1 \in L^\infty(0, T; L^2(\Omega; H^1_0(M))).$$
• Thus, with respect to the fast variable $y$, $v$ vanishes outside soft part $B$ and $u_1$ vanishes outside the stiff part $M$ taking care of the two features.
Remarks

- Thus, with respect to the fast variable $y$, $v$ vanishes outside soft part $B$ and $u_1$ vanishes outside the stiff part $M$ taking care of the two features.

- We now state a weak formulation for the unknowns $\tilde{u}$, $v$ and $u_1$. 

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• We now state a weak formulation for the unknowns $\tilde{u}$, $v$ and $u_1$.

• We then represent $u_1$ in terms of $\tilde{u}$ leading to a two-scale homogenized system for $\tilde{u}$ and $v$. 
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• We also derive appropriate initial conditions so that the two-scale system is well-posed.
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• Thus, with respect to the fast variable $y, v$ vanishes outside soft part $B$ and $u_1$ vanishes outside the stiff part $M$ taking care of the two features.

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• We then represent $u_1$ in terms of $\tilde{u}$ leading to a two-scale homogenized system for $\tilde{u}$ and $v$.

• We also derive appropriate initial conditions so that the two-scale system is well-posed.

• Deriving a one-scale system (homogenized) remains open, probably, it is not possible, whereas such a decomposition is possible in an elliptic system.
2-scale system for $\tilde{u}$, $v$ and $u_1$.

Find $\tilde{u}$, $v$ and $u_1$ in appropriate spaces such that

\[
\int_{\Omega_T} \int_Y (\tilde{u} + v) (\bar{u} + \bar{v})'' + \int_{\Omega_T} \int_Y (\tilde{u} + v) (\bar{u} + \bar{v})
\]
\[
+ \int_{\Omega_T} \int_B A(x, y) \nabla_y v \cdot \nabla_y \bar{v}
\]
\[
+ \int_{\Omega_T} \int_M A(x, y)(\nabla_x \tilde{u} + \nabla_y u_1)(\nabla_x \bar{u} + \nabla_y \bar{u}_1)
\]
\[
= \int_{\Omega_T} f \bar{u} + \int_{\Omega_T} \int_Y \tilde{f} \bar{v}
\]

for the appropriate test functions $\bar{u}$, $\bar{v}$ and $\bar{u}_1$. Here $f$ and $\tilde{f}$ are, respectively, the $L^2$ weak-limit and two-scale limit of the data $f_\varepsilon$. 
Representation of $u_1$

**Proposition**

The solution $u_1$ can be given in terms of $\tilde{u}$ and solutions to cell problem as

$$u_1(t, x, y) = \sum \frac{\partial \tilde{u}}{\partial x_i} w_i(x, y),$$

where $w_i$ satisfies for a.e. $x \in \Omega$:

\[
\begin{align*}
\quad -\text{div}_y (A(x, y)(\nabla_y w_i(x, \cdot) + e_i)) &= 0 \quad \text{in} \quad M \\
\quad w_i(x, \cdot) &\in H^1_0(M)
\end{align*}
\]

Here $\{e_i, 1 \leq i \leq n\}$ is the canonical basis of $\mathbb{R}^n$. 
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$$\begin{cases} 
- \mathrm{div}_y (A(x, y)(\nabla_y w_i(x, \cdot) + e_i)) = 0 \text{ in } M \\
   w_i(x, \cdot) \in H^1_{0#}(M)
\end{cases}$$

Here $\{e_i, 1 \leq i \leq n\}$ is the canonical basis of $\mathbb{R}^n$.

Introducing the homogenized matrix

$$A^*(x) = \int_M A(x, y)(\nabla_y w_j + e_j)(\nabla_y w_i + e_i) dy,$$

we may eliminate $u_1$. 
2-scale system in terms of $\tilde{u}$ and $v$.

**Theorem**

The limits $\tilde{u}$ and $v$ satisfies

$$\tilde{u} \in L^\infty(0, T; H^1_0(\Omega)), \ 	ilde{u}' \in L^\infty(0, T; L^2(\Omega)),
$$

$$v \in L^\infty(0, T; L^2(\Omega; H^1_0(B))), \ v' \in L^\infty(0, T; L^2(\Omega; L^2(B))),$$

and

$$\int_{\Omega_T} \int_Y (\tilde{u} + v) (\tilde{u} + \bar{v})'' + \int_{\Omega_T} \int_Y (\tilde{u} + v) (\tilde{u} + \bar{v})$$

$$+ \int_{\Omega_T} \int_B A(x, y) \nabla_y v \cdot \nabla_y \bar{v} + \int_{\Omega_T} A^*(x) \nabla \tilde{u} \cdot \nabla \bar{u}$$

$$= \int_{\Omega_T} f \tilde{u} + \int_{\Omega_T} \int_Y \bar{f} \bar{v} \quad (12)$$

for all smooth functions $\bar{u}, \bar{v}$.
Two-scale Homogenized System (Strong Form)

Find \( \tilde{u} \in L^\infty(0, T; H^1_0(\Omega)) \), \( v \in L^\infty(0, T; L^2(\Omega; H^1_0(B))) \) s.t.

\[
(\tilde{u} + v)'' + (\tilde{u} + v) - \text{div}_y(\chi_B A(x, y) \nabla_y v) - \text{div}_x(A^*(x) \nabla \tilde{u}) = f + \bar{f}
\]

\[\tilde{u}(0, x) = \int_M u^0(x, y) \, dy \quad \tilde{u}'(0, x) = \int_M u^1(x, y) \, dy \quad \text{in } \Omega,\]

\[v(0, x, y) = u^0(x, y) - \int_M u^0(x, y) \, dy\]

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- Need to interpret the meaning of $\tilde{u}(0)$, $v(0)$ and $\tilde{u}'(0)$, $v'(0)$ and
Two-scale Homogenized System (Strong Form)

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\]

- Need to interpret the meaning of \( \tilde{u}(0) \), \( v(0) \) and \( \tilde{u}'(0) \), \( v'(0) \) and then, identify them.
Meaning of Initial Values

- Meaning to $v_0(0) = (\tilde{u} + v)(0)$, $v_0'(0) = (\tilde{u} + v)'(0)$
Meaning of Initial Values

- Meaning to $v_0(0) = (\tilde{u} + v)(0)$, $v'_0(0) = (\tilde{u} + v)'(0)$
- This can be interpreted properly using the facts

$$\tilde{u} \in L^\infty_t H^1_0(\Omega), \tilde{u}' \in L^\infty_t L^2(\Omega), \tilde{u}'' \in L^\infty_t H^{-1}(\Omega)$$

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and

$$v \in L_t^\infty L_x^2 H^1_y, v' \in L_t^\infty L_x^2 L_y^2, v'' \in L^\infty H^{-1}(\Omega \times Y).$$
Meaning of Initial Values

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\]

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\[
v \in L^\infty_t L^2_x H^1_y, \ v' \in L^\infty_t L^2_x L^2_y, \ v'' \in L^\infty_t H^{-1}(\Omega \times Y).
\]

• The above regularity implies that \( \tilde{u}, v \in C([0, T], L^2(\Omega \times Y)) \) and \( \tilde{u}', v' \in C([0, T], H^{-1}(\Omega \times Y)) \) and hence

\[
\tilde{u}(0), v(0) \in L^2(\Omega \times Y), \ \tilde{u}'(0), v'(0) \in H^{-1}(\Omega \times Y)
\]
Identification of Initial Conditions

- Using the boundedness of the initial conditions $u_\varepsilon^0$ and $u_\varepsilon^1$, let
  
  $u_\varepsilon^0 \xrightarrow{2-s} u^0(x, y), \quad u_\varepsilon^1 \xrightarrow{2-s} u^1(x, y)$.

- Using the strong convergence of the extension $\tilde{u}_\varepsilon$, it is easy to find the initial condition for $\tilde{u}$.

- However, more work to be done to identify the limit for $v(0), v'(0)$. 
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• Using the strong convergence of the extension $\tilde{u}_\varepsilon$, the it is easy to find the initial condition for $\tilde{u}, \tilde{u}'(0)$.

• However, more work to be done to identify the limit for $v(0), v'(0)$. 
Existence

• The above equation defines a hyperbolic system with appropriate elliptic part. Let \( X = H^1_0(\Omega) \), \( Z = L^2(\Omega; H^1_0(B)) \). We have \((\tilde{u}, v) \in L^\infty(0, T; X) \times L^\infty(0, T; Z)\).
Existence

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• **Elliptic bilinear form:** Let $A : X \times Z \to \mathbb{R}$, $U = (u, v) \in X \times Z$ defined by

$$A(U_1, U_2) = \int_{\Omega} u_1 u_1 + \int_{\Omega \times B} v_1 v_1 + \int_{\Omega} A^*(x) \nabla u_1 \cdot \nabla u_2$$

$$+ \int_{\Omega \times B} A(x, y) \nabla_y v_1 \cdot \nabla_y v_2$$
• The norm $\|U\|_{X \times Z}^2 = \|u\|_{H^1_0(\Omega)}^2 + \|\nabla_y v\|_{L^2(\Omega \times B)}^2 + \|v\|_{L^2(\Omega \times B)}^2$ is equivalent to $\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_y v\|_{L^2(\Omega \times B)}^2$. 
The norm $\|U\|_{X \times Z}^2 = \|u\|_{H^1_0(\Omega)}^2 + \|\nabla_y v\|_{L^2(\Omega \times B)}^2 + \|v\|_{L^2(\Omega \times B)}^2$ is equivalent to $\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_y v\|_{L^2(\Omega \times B)}^2$.

Clearly $A$ is continuous and

$$A(U, U) = \int_{\Omega} |u_1|^2 + \int_{\Omega \times B} |v_1|^2$$

$$+ \int_{\Omega} A^*(x) \nabla u \cdot \nabla u + \int_{\Omega \times B} A(x, y) \nabla_y v \cdot \nabla_y v$$

$$\geq C \left[ \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla_y v|^2 \right] \geq C_1 \|U\|_{X \times Z}^2$$

The existence of the Hyperbolic system could be worked out.
The norm $\|U\|_{X \times Z}^2 = \|u\|_{H_0^1(\Omega)}^2 + \|\nabla_y v\|_{L^2(\Omega \times B)}^2 + \|v\|_{L^2(\Omega \times B)}^2$ is equivalent to $\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_y v\|_{L^2(\Omega \times B)}^2$.

Clearly $A$ is continuous and

$$A(U, U) = \int_{\Omega} |u_1|^2 + \int_{\Omega \times B} |v_1|^2$$

$$+ \int_{\Omega} A^*(x) \nabla u \cdot \nabla u + \int_{\Omega \times B} A(x, y) \nabla_y v \cdot \nabla_y v$$

$$\geq C \left[ \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla_y v|^2 \right] \geq C_1 \|U\|_{X \times Z}^2$$

The existence of the Hyperbolic system could be worked out.
Remarks

• We do not know how to separate $\tilde{u}$ and $v$ so that we have a complete homogenized equation for the macro variable alone, probably it may not be possible.
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- A complete decoupling is possible in an elliptic system.
Decoupling in elliptic Problem

The elliptic two scale system is given by

\[
\int_{\Omega} \int_{\mathcal{Y}} (\tilde{u} + v) (\bar{u} + \bar{v}) + \int_{\Omega} \int_{\mathcal{B}} A(x, y) \nabla_y v \cdot \nabla_y \bar{u} \\
+ \int_{\Omega} A^*(x) \nabla \tilde{u} \cdot \nabla \bar{u} = \int_{\Omega} f \bar{u}
\]

(13)
Decoupling in elliptic Problem

The elliptic two scale system is given by

\[
\int_{\Omega} \int_{Y} (\tilde{u} + v) (\bar{u} + \bar{v}) + \int_{\Omega} \int_{B} A(x, y) \nabla_y v \cdot \nabla_y \bar{v} + \int_{\Omega} \int_{B} A^*(x) \nabla \hat{u} \cdot \nabla \bar{u} = \int_{\Omega} f \bar{u}
\]

(13)

Introduce, \( \hat{v}(x, y) \in L^\infty(\Omega \times B) \) as the solution to the cell problem for a.e. \( x \in \Omega; \)

\[
\hat{v}(x, \cdot) - \text{div}_y A(x, \cdot) \nabla u \hat{v}(x, \cdot) = 1 \quad \text{in B}
\]
Decoupling in elliptic Problem

The elliptic two scale system is given by

\[
\int_{\Omega} \int_{Y} (\tilde{u} + v)(\bar{u} + \bar{v}) + \int_{\Omega} \int_{B} A(x, y) \nabla_y v \cdot \nabla_y \bar{v} + \int_{\Omega} A^*(x) \nabla \tilde{u} \cdot \nabla \tilde{u} = \int_{\Omega} f \bar{u}
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\]

Then, \( v \) is given by \( v(x, y) = (f(x) - \tilde{u}(x))\hat{v}(x, y) \) and \( \tilde{u} \) satisfies the homogenized equation; \( \tilde{u} \in H^1_0(\Omega) \)

\[
m(x) u(x) - \text{div} A^*(x) \nabla u(x) = m(x) f(x) \quad \text{in} \quad \Omega,
\]

where \( m(x) = \int_{B} \hat{v}(x, y) dy. \)
Decoupling in Hyperbolic System Problem

Introduce $\hat{u}(t, x), \hat{v}(t, x, y)$ as follows:

\[
\begin{cases}
\text{Find } \hat{u} \in L^\infty(0, T; H^1_0(\Omega)), \hat{u}' \in L^\infty(0, T; L^2(\Omega)) \text{ satisfying} \\
\hat{u}'' + \hat{u} - \text{div}_x (A^*(x)\nabla \hat{u}) = f \text{ in } \Omega_T \\
\hat{u}(0) = 0 = \hat{u}'(0).
\end{cases}
\tag{14}
\]
Decoupling in Hyperbolic System Problem

Introduce $\hat{u}(t, x)$, $\hat{v}(t, x, y)$ as follows:

Find $\hat{u} \in L^\infty(0, T; H_0^1(\Omega))$, $\hat{u}' \in L^\infty(0, T; L^2(\Omega))$ satisfying

$$\hat{u}'' + \hat{u} - \text{div}_x (A^*(x) \nabla \hat{u}) = f \text{ in } \Omega_T$$

$$\hat{u}(0) = 0 = \hat{u}'(0).$$

(14)

and with $x \in \Omega$ as parameter

Find $\hat{v} \in L^\infty(0, T; H_0^1(B))$, $\hat{v}' \in L^\infty(0, T; L^2(B))$ satisfying

$$\hat{v}'' + \hat{v} - \text{div}_y (A(x, y) \nabla_y \hat{v}) = f \text{ in } (0, T) \times B$$

$$\hat{v}(0, x, y) = u^{00}(x, y)$$

$$\hat{v}'(0, x, y) = u^{11}(x, y).$$

(15)
Decoupling in Hyperbolic System Problem

• Indeed $\hat{u} + \hat{v}$ satisfies the two-scale system. By uniqueness, the two-scale limit of the inhomogenized equation is given by

$$v_0 = \tilde{u} + v = \hat{u} + \hat{v},$$

where $\hat{u}, \hat{v}$ are given as in (14) and (15) respectively.
Decoupling in Hyperbolic System Problem

• Indeed $\hat{u} + \hat{v}$ satisfies the two-scale system. By uniqueness, the two-scale limit of the inhomogenized equation is given by

$$v_0 = \tilde{u} + v = \hat{u} + \hat{v}, \quad (16)$$

where $\hat{u}, \hat{v}$ are given as in (14) and (15) respectively.

• Though, we have a complete representation of the weak limit via $\hat{u}$ and $\hat{v}$, the equation (14) cannot be treated as a complete homogenized (one-scale) system as $\hat{u}$ do not capture the initial values.
Main Theorem

**Theorem (Homogenization)**

Let the given data $f_\varepsilon, u_\varepsilon^0, u_\varepsilon^1, a_\varepsilon$ satisfy the assumptions be given. Let $u_\varepsilon$ be the unique solution to the problem $(P_\varepsilon)$. Then,

$$u_\varepsilon \xrightarrow{2s} \tilde{u}(t, x) + v(t, x, y),$$

where the pair $(\tilde{u}, v) \in L^\infty(0, T; H_0^1(\Omega)) \times L^2(\Omega_T; H_0^1(B))$ is the unique solution of the coupled system described earlier.
The other two regimes; \( \alpha := \lim_{\varepsilon \to 0} \frac{\alpha \varepsilon}{\varepsilon} = 0 \) or \( \alpha = +\infty \).

- The case \( \alpha = 0 \); Here the limit problem is

\[
(1 - |B|)(u'' + u) - \text{div}_x A^*(x) \nabla u = \int_M f_0 \, dy, \quad \text{in } \Omega_T,
\]

with the same initial conditions as earlier.
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\[
(1 - |B|)(u'' + u) - \text{div}_x A^*(x) \nabla u = \int_M f_0 \, dy, \quad \text{in } \Omega_T,
\]

with the same initial conditions as earlier.

- The contribution of the soft part \( B_{\varepsilon} \) in the homogenized equation is seen through the measure of \( B \) in the final macroscopic equation.
The case $\alpha = \infty$

- There is no contribution from the soft material. The homogenized equation is

$$u'' + u - \text{div}_x A^* (x) \nabla u = \int_Y f_0 \, dy, \quad \text{in } \Omega_T,$$

with the same initial conditions as in the critical case.

The case \( \alpha = \infty \)

- There is no contribution from the soft material. The homogenized equation is

\[
\frac{d^2 u}{dx^2} + u - \text{div}_x A^*(x) \nabla u = \int_Y f_0 \, dy, \quad \text{in } \Omega_T,
\]

with the same initial conditions as in the critical case.
The case $\alpha = \infty$

- There is no contribution from the soft material. The homogenized equation is

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Thank You!

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