1. Show that if p is an odd prime number then
$$\left\{\left(\frac{p-1}{2}\right)!\right\}^2 \equiv (-1)^{(p+1)/2} \mod p$$

2. (a) Show that the following numbers are algebraic over $\mathbb{Q}$, determine the minimal polynomial of each number and find all of its conjugates over $\mathbb{Q}$:
$$2-3i; \quad \sqrt{2+\sqrt{2}}; \quad \frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{2}$$
   (b) Which among the numbers given in (a) are algebraic integers?

3. Prove the following properties of norms: Let K be a number field and let $\alpha \in \mathcal{O}_K$.
   (a) Prove that $\mathrm{Nm}(\alpha) \in \mathbb{Z}$.
   (c) Prove that $\mathrm{Nm}(\alpha) = \pm 1$ if and only if α is a unit in $\mathcal{O}_K$.
   (b) If $\mathrm{Nm}(\alpha)$ is a rational prime, then α is irreducible in $\mathcal{O}_K$.

4. Give an example of a Gaussian number $\gamma = a + bi$ such that $\mathrm{Nm}(\gamma) = 1$ but γ is not an algebraic integer.

5. Compute the norm of a generic element $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ in $\mathbb{Q}(\sqrt[3]{2})$. Show that it is in $\mathbb{Q}$.

6. In a number field K, show that if α and β are roots of a monic polynomial in $\mathbb{Z}[x]$ and $\alpha, \beta \in K$ then $\alpha + \beta$ and $\alpha\beta$ are also roots of a monic polynomial in $\mathbb{Z}[x]$. In other words, prove that the ring of integers is indeed a ring!

7. Let d be a squarefree integer. Prove that the ring of integers of $\mathbb{Q}(\sqrt{d})$ is
$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}], & \text{if } d \equiv 2,3 \pmod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right], & \text{if } d \equiv 1 \pmod 4 \end{cases}$$

8. Show that the ring of algebraic integers of $\mathbb{Q}(\sqrt{10})$ is not a unique factorization domain.

9. For each of the following irreducible polynomials, let α be a root and $K = \mathbb{Q}(\alpha)$. Compute $\mathcal{O}_K$ and factorizations of $(2), (3), (5), (7)$ in $\mathcal{O}_K$.
$$f(x) = x^2 + 31; \quad f(x) = x^2 - 29; \quad f(x) = x^3 + x - 1$$

10. Decompose $33 + 11\sqrt{-7}$ into irreducible integral elements of $\mathbb{Q}(\sqrt{-7})$.

11. Show that for any ideal I of $\mathcal{O}_K$, $\mathcal{O}_K/I$ is finite.

12. Show that $(2, 1+\sqrt{-5})$ and $(2, 1-\sqrt{-5})$ are the same ideal in $\mathbb{Z}[\sqrt{-5}]$.

13. Prove that every ideal of a Dedekind domain can be generated by two elements.