PROBLEMS MODULAR FORMS CIMPA 2023

MAR CURCÓ IRANZO, LAURA GEATTI, VALENTIJN KAREMAKER

1. List 2

Problem 1. *In this exercise you will work on the structure of the graded algebra of modular forms.*

(a) *Let $f \in M_k$ and $g \in M_l$. Show that $f \cdot g$ defines a modular form of weight $k+l$. Deduce that $M = \oplus_k M_k$ is a graded algebra.*

(b) *Show that the map $X \mapsto E_4; Y \mapsto E_6$ defines an isomorphism between $\mathbb{C}[X,Y]$ and M. (So we will make the identification with the polynomial algebra in $E_4, E_6 : M = \mathbb{C}[E_4, E_6]$.)*

Problem 2. *Spaces of modular forms.*

(a) *Show that $M_{14} = \mathbb{C}E_{14}$, $S_{14} = \{0\}$ and $E_{14} = E_6 E_8 = E_6 E_4^2$. (Hint: for the cusp form spaces, use the isomorphism $f \mapsto \Delta f$ between the spaces M_k and S_{k+12}.)*

(b) *Recall the definition of Dedekind's eta function $\eta(z) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$. In Exercise 6(d) of the 1st training session we saw that $\eta(z)^{24}$ is a modular form of weight 12. Use this fact to show that the modular discriminant $\Delta(z)$ and $\eta(z)^{24}$ are equal.*

Problem 3. *The goal of this exercise is to find an expression for the Fourier expansion of $G_k(z)$ in a different way that the one presented in the lectures.*

(a) *Show that*

$$\frac{\pi}{tan(\pi z)} = -2\pi i \left(\frac{1}{2} + \sum_{r=1}^{\infty} q^r \right).$$

(Hint: use the complex exponential trigonometric identities to find the q-expansion of the function on the left hand side. Recall that $\frac{1}{q-1} = \sum_{r=0}^{\infty} q^r$.)

(b) *Use Euler's identity $\pi \frac{cos(\pi z)}{sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right)$ to show that*

$$\sum_{n=1}^{\infty} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^r.$$

(Hint: consider the $(k-1)$th-derivative of the formula obtained in (a)).

(c) *Use the identity $\zeta(k) = -\frac{(2\pi i)^k B_k}{2 \cdot k!}$ (see bonus problem 6) to show that*

$$G_k(z) = \frac{2\pi i}{(k-1)!} \left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n \right).$$

Date: January 11, 2023.

$B_0 = 1$	$B_6 = \frac{1}{42}$	$B_{12} = \frac{-691}{2730}$	$B_{18} = \frac{43867}{798}$
$B_1 = \frac{-1}{2}$	$B_7 = 0$	$B_{13} = 0$	$B_{19} = 0$
$B_2 = \frac{1}{6}$	$B_8 = \frac{-1}{30}$	$B_{14} = \frac{7}{6}$	$B_{20} = \frac{-174611}{330}$
$B_3 = 0$	$B_9 = 0$	$B_{15} = 0$	$B_{21} = 0$
$B_4 = \frac{-1}{30}$	$B_{10} = \frac{-1}{30}$	$B_{16} = \frac{-3617}{510}$	$B_{22} = \frac{854513}{138}$
$B_5 = 0$	$B_{11} = 0$	$B_{17} = 0$	$B_{23} = 0$

FIGURE 1. The first Bernoulli numbers.

(Hint: split the expression of $G_k(z)$ into two sums, one with the terms $m = 0$ and one with the terms $m \neq 0$.)

Problem 4. *Let $n \in \mathbb{Z}$, $n > 0$. Recall the q-expansion: $E_k(z) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n)\, q^n$.*

(a) Use Figure 1 to verify that the q-expansion of E_k for $k \leq 14$ is the following:

$$E_4(z) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, \qquad E_6(z) = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n,$$

$$E_8(z) = 1 + 480 \sum_{n\geq 1} \sigma_7(n) q^n, \qquad E_{10}(z) = 1 - 264 \sum_{n\geq 1} \sigma_9(n) q^n,$$

$$E_{12}(z) = 1 + \frac{6520}{691} \sum_{n\geq 1} \sigma_{11}(n) q^n, \qquad E_{14}(z) = 1 - 24 \sum_{n\geq 1} \sigma_{13}(n) q^n.$$

(b) Prove that $\sigma_7(n) \equiv \sigma_3(n) \pmod{120}$.

(c) Prove that $11\sigma_9(n) = -10\sigma_3(n) + 21\sigma_5(n) + 5040 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_5(n-j)$.

(d) Use the same techniques to find expressions for σ_{13} in terms of σ_3 and σ_9, and in terms of σ_5 and σ_7.

(e) Use the previous parts to write σ_{13} in terms of σ_3 and σ_5.

Problem 5. *Congruences on Ramanujan's tau function.*

(a) Prove that

$$E_6^2 = E_{12} - \frac{762048}{691} \Delta.$$

(b) Using the factorisations $504 = 2^3 3^2 7$, $65620 = 2^4 3^2 5^7 13$, $762048 = 2^6 3^5 7^2$, deduce that

$$756\tau(n) = 65\sigma_{11}(n) + 691\sigma_5(n) - 252 \cdot 691 \sum_{j=1}^{n-1} \sigma_5(j)\sigma_5(n-j).$$

Deduce the Ramanujan's congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

Problem 6. *From the dimension formula for the spaces of modular forms we can see that the only integers k for which $\dim(S_k) = 2$ are $k \in \{24, 28, 30, 32, 34, 36, 38\}$.*

(a) *Choose your favourite k from the list above and use the commands* `mfinit` *and* `mfbasis` *in* `PARI/GP` *to obtain a basis $\{f_1, f_2\}$ for the space S_k with the chosen k.*
(b) *Use the properties of Hecke operators given in the lectures to compute $T_2 f_1$ and $T_2 f_2$. That is, write the cusp forms $T_2 f_1$ and $T_2 f_2$ in terms of the basis $\{f_1, f_2\}$.*
(c) *Use the command* `mfhecke` *to check your computations of part (b).*

Problem 7. *(Bonus) In this exercise you will prove a relation between Bernoulli Numbers and the Riemann zeta function. Recall that Bernoulli numbers can be defined as the rational numbers B_k for $k \geq 0$ given by the equation*

$$\frac{t}{exp(t)-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k \in \mathbb{Q}[[t]].$$

(a) *Show that for all $|z| < 1$ we have*

$$\pi z \frac{cos(\pi z)}{sin(\pi z)} = \sum_{k \geq 0} (2\pi i)^k \frac{B_k}{k!} z^k.$$

(b) *Use Euler's identity $\pi \frac{cos(\pi z)}{sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n}\right)$ to prove that*

$$\pi z \frac{cos(\pi z)}{sin(\pi z)} = 1 - 2 \sum_{k \geq 2, even} \zeta(k) z^k.$$

(c) *Deduce that*

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2 \cdot k!}.$$

Problem 8. *(Bonus) Given a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, where $\omega_1/\omega_2 \in H$, we define the j-invariant $j(\Lambda)$ of Λ as the value $j(\omega_1/\omega_2) := 1728 \frac{E_4(\omega_1/\omega_2)^3}{E_4(\omega_1/\omega_2)^3 - E_6(\omega_1/\omega_2)^2}$.*

(a) *Show that $j(z)$ is a modular function.*

The q-expansion of j looks like $j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$.

(b) *Show that $j(i) = 1728$ and $j(\rho) = 0$ (where $\rho = \exp(2\pi i/3)$).*
(c) *Let $z \in \mathcal{D}$ (see Problem 1 of List 1 for the definition of $\mathcal{D}$). Prove the following statement: if z lies on the boundary of $\mathcal{D}$ or $Re(z) = 0$, then $j(z) \in \mathbb{R}$.*
(d) *Show that $j : \mathrm{SL}_2(\mathbb{Z})\backslash H \to \mathbb{C}$ given by $j([z]) := j(z)$ is well-defined and prove that j is bijective. (Here $[z]$ denotes the orbit of z under the action of $\mathrm{SL}_2(\mathbb{Z})$.) Conclude that the j-invariant gives a bijection*

$$\{lattices\ in\ \mathbb{C}\}/(homothety) \to \mathbb{C}.$$

(Hint: Recall that there is a bijection between the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash H$ and $\{lattices\ in\ \mathbb{C}\}/(homothety)$.)
(e) *Prove the converse to part (c).*