PROBLEMS MODULAR FORMS CIMPA 2023

MAR CURCÓ IRANZO, LAURA GEATTI, VALENTIJN KAREMAKER

1. LIST 1

Problem 1. *Let $\mathcal{D} := \{z \in H : |z| \geq 1$ and $|Re(z)| \leq 1/2\}$ be the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ described in the lectures. Prove the following statements.*

(a) *If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is such that $\gamma(\tau_1) = \tau_2$ for some $\tau_1, \tau_2 \in \mathcal{D}$, then $c \in \{-1, 0, 1\}$.*

(b) *If $\gamma = \left(\begin{smallmatrix} a & b \\ 1 & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is such that $\gamma(\tau_1) = \tau_2$ for some $\tau_1, \tau_2 \in \mathcal{D}$, then $d \in \{-1, 0, 1\}$*

(c) *If $\gamma = \left(\begin{smallmatrix} a & b \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ then $b = -1$. Moreover, if γ is such that $\gamma(\tau_1) = \tau_2$ for some $\tau_1, \tau_2 \in \mathcal{D}$, then $a \in \{-1, 0, 1\}$. Moreover, for each $a \in \{-1, 0, 1\}$, describe which are the possible τ_1 and τ_2 (note that $\tau_1 = \tau_2$ is admissible).*

Problem 2. *In this exercise you will prove that the matrices $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ generate the group $\mathrm{SL}_2(\mathbb{Z})$.*

(a) *Consider $\Gamma' := \langle T, S \rangle \subseteq \mathrm{SL}_2(\mathbb{Z})$. Given any $z \in H$, show that there is a matrix $\gamma \in \Gamma'$ such that $Im(\gamma z)$ is maximal in the orbit of z under the group Γ'. (Hint: use the formula $Im(\gamma z) = \frac{Im(z)}{|cz+d|^2}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and mimic the arguments given in the lectures.)*

(b) *Prove the classification of the stabilizers given in the lectures. That is, if $I(z)$ is the stabilizer of $z \in \mathcal{D}$ under $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \backslash \{\pm Id\}$, show that*
   1. *if $z \in int(\mathcal{D})$, then $I(z) = \{Id\}$.*
   2. *if $z = i$, then $I(z) = \{Id, S\}$.*
   3. *if $z = \rho = e^{2\pi i/3}$, then $I(z) = \{Id, ST, (ST)^2\}$.*
   4. *if $z = -\overline{\rho} = e^{\pi i/3}$, then $I(z) = \{Id, TS, (TS)^2\}$.*

(c) *Use part (a) and (b) to show that $\Gamma' = \mathrm{SL}_2(\mathbb{Z})$. (Hint: consider z a point in the interior of the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$.)*

Problem 3. *For $k > 2$ an integer, we defined the function $G_k(\Lambda)$ on lattices of $\mathbb{C}$ and we considered $G_k(z)$ the expression of $G_k(\Lambda)$ as a function defined on H. On the other hand we also defined the function $E_k(z)$ on the complex upper-half plane H.*

(a) *Show that $G_k(z) = 2\zeta(k) \cdot E_k(z)$, where $\zeta(k)$ is Riemann's Zeta function $\zeta(k) = \sum_{r \geq 1} \frac{1}{r^k}$. (Hint: given $(m, n) \neq (0, 0)$ in $\mathbb{Z}^2$, consider $r = \gcd(m, n)$).*

(b) *Use Problem 2 to show that $E_k(z)$ transforms as a modular form of weight k; that is,*

$$E_k(\gamma z) = (cz + d)^k E_k(z)$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ by checking it for S and T.

Date: January 11, 2023.

Problem 4. *Zeroes of modular forms.*

(a) *Let $\rho = exp(2\pi i/3)$. Show that $E_4(\rho) = 0$. (Hint: $E_4(-1/z) = z^4 E_4(z)$.)*

(b) *Show that $E_6(i) = 0$.*

(c) *Use the Valence formula to show that E_4 has a simple zero at ρ and no other zero in H and E_6 has a simple zero at i and no other zero in H.*

Problem 5. *In this exercise you will show that the only modular forms of weight zero are constant functions.*

(a) *Show that there exists $C \in \mathbb{R}_{>0}$ such that any element in H is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to some $z \in H$ with $Im(z) \geq C$. (You can take e.g. $C = \sqrt{3}/2$.)*

(b) *Deduce that if $f : H \to \mathbb{C}$ is a cusp form of weight 0, then $|f|$ attains a maximum on H.*

(c) *Conclude that the space of modular forms of weight zero consists exactly of the constant functions $H \to \mathbb{C}$. (Hint: use the maximum modulus principle.)*

Problem 6 (Bonus). *Recall the definition of the Eisenstein series $E_k(z) = \frac{1}{2}\sum_{(c,d)\in\mathbb{Z}^2, \gcd(c,d)=1} \frac{1}{(cz+d)^k}$. For $k = 2$ this is not a modular function, but also satisfies some transformation properties w.r.t. the action of $\mathrm{SL}_2(\mathbb{Z})$.*

(a) *Show that E_2 satisfies*

$$E_2(z+1) = E_2(z),$$
$$z^{-2}E_2\left(-\frac{1}{z}\right) = E_2(z) + \frac{12}{2\pi i z},$$
$$(cz+d)^{-2}E_2\left(\frac{az+b}{cz+d}\right) = E_2(z) + \frac{12c}{2\pi i(cz+d)}.$$

You can use without proof the two following identities:

$$\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) = 0,$$
$$\sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) = -\frac{2\pi i}{z}.$$

We define Dedekind's eta function as $\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, where $q = \exp^{2\pi i z}$.

(b) *Show that $\frac{1}{2\pi i}\frac{d}{dz}log(\eta(z)) = \frac{1}{24}E_2(z)$.*

(c) *Show that the function $\eta(z)$ satisfies*

$$\eta(z+1) = \exp\left(\frac{1}{24}\right)\eta(z),$$
$$\eta\left(\frac{-1}{z}\right) = \sqrt{-iz}\,\eta(z),$$
$$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon_\eta(a,b,c,d)\sqrt{-i(cz+d)}\,\eta(z),$$

where $\epsilon_\eta(a,b,c,d)$ is some 24th root of unity.

(d) *Conclude that $\eta(z)^{24}$ is a modular form of weight 12.*