1. Let K be a number field, and let $\mathrm{Cl}(K)$ be the class group of K. Prove that the relation on ideals I and J of $\mathcal{O}_K$ defined as $I \sim J$ iff there exists $\alpha, \beta \in \mathcal{O}_K$ such that $\alpha I = \beta J$. Prove that this is an equivalence relation. Prove the set of equivalence classes under $\sim$ on ideals of $\mathcal{O}_K$ equipped with the operation induced by ideal multiplication is an (abelian) group.

2. Let $K = \mathbb{Q}(\gamma)$, where γ is the root of a monic irreducible polynomial $f(x)$ of degree d. Show that there are at most d embeddings of $K \hookrightarrow \mathbb{C}$ that fix $\mathbb{Q}$ pointwise.

3. Prove that the numbers $\zeta_3 \cdot \sqrt[3]{2}$ and $\zeta_3^2 \cdot \sqrt[3]{2}$ are *not* in $\mathbb{Q}(\sqrt[3]{2})$.

4. Show that $\mathrm{Gal}(\mathbb{Q}\left(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}\right) = S_3$, where S_3 is the symmetric group on 3 letters.

5. If K is a normal extension of $\mathbb{Q}$, and $\mathrm{Gal}(K/\mathbb{Q})$ denotes the Galois group. Let H be a subgroup of $\mathrm{Gal}(K/\mathbb{Q})$, define

$$K^H = \{\alpha \in K \mid \sigma(\alpha) = \alpha \quad \forall \sigma \in H\}.$$

Prove that K^H is a field.

6. Let K be a normal extension of $\mathbb{Q}$, and let $\mathfrak{P}$ be an unramified prime ideal in $\mathcal{O}_K$ lying above a rational prime p. For a prime ideal $\mathfrak{P}'$ also lying above p, recall that there is an element $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\sigma(\mathfrak{P}) = \mathfrak{P}'$. Show that $\mathrm{Frob}_{\mathfrak{P}} = \sigma \mathrm{Frob}_{\mathfrak{P}'} \sigma^{-1}$.

7. Let $K = \mathbb{Q}(\sqrt[3]{19}, \zeta_3)$.
   (a) What is the Galois group over $\mathbb{Q}$?
   (b) How does (3) factor in $\mathcal{O}_K$?
   (c) Compute e, f, and g for $p = 3$.
   (d) Can you compute the decomposition group and inertia group of a prime above 3?

8. Determine the decomposition group and the inertia group for the primes above 2 in $\mathbb{Q}(\zeta_{23})$.